

# 3D Orthogonal Curvilinear Coordinates Systems

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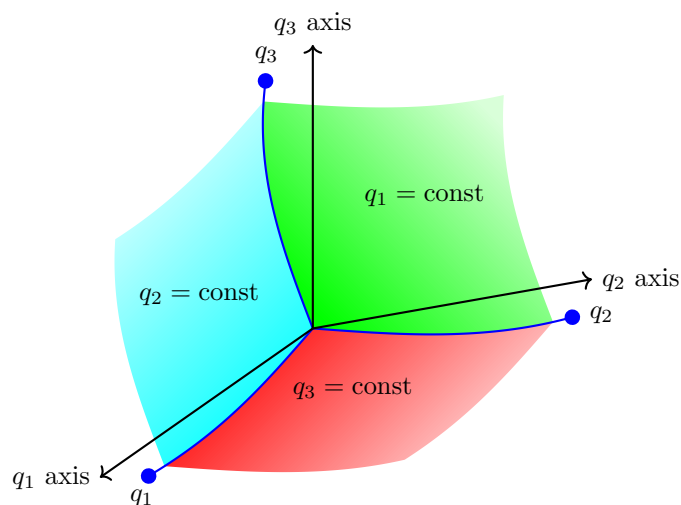
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# 1 Introduction

A curvilinear coordinate system is a coordinate system composed of intersecting surfaces. If the intersections are all at right angles, then the curvilinear coordinates are said to form an orthogonal coordinate system. If not, they form a skew coordinate system. An orthogonal coordinate system is a system of curvilinear coordinates in which each family of surfaces intersects the others at right angles. Orthogonal coordinates therefore satisfy the additional constraint that

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

In a curvilinear coordinate system, the position of a point is specified by a set of coordinates, and each coordinate is associated with a curve or surface. The choice of curves or surfaces is often motivated by the symmetry of the problem being analyzed. The primary advantage of curvilinear systems is that they simplify the mathematical representation of physical systems with specific symmetries.



## 2 Spherical Polar Coordinates

$$(x, y, z) \longleftrightarrow (r, \theta, \phi)$$

$$x = r \sin \theta \cos \phi$$

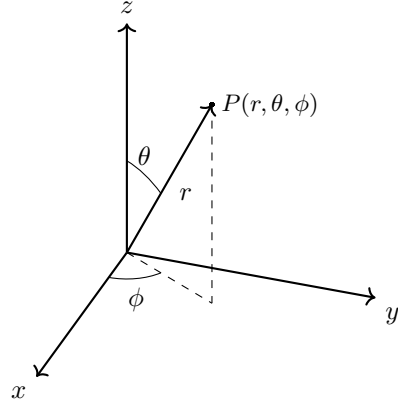
$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$r \in [0, \infty), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi)$$

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$= r \sin \theta \cos \phi \hat{\mathbf{r}} + r \sin \theta \sin \phi \hat{\boldsymbol{\theta}} + r \cos \theta \hat{\boldsymbol{\phi}}$$



### Scale Factors

$$\begin{aligned} h_r &= \left| \frac{\partial \mathbf{r}}{\partial r} \right| = \sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta + \cos^2 \theta} \\ &= 1 \end{aligned}$$

$$\begin{aligned} h_\theta &= \left| \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta} \\ &= \sqrt{r^2 \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^2 \theta} \\ &= \sqrt{r^2 (\cos^2 \theta + \sin^2 \theta)} \\ &= \sqrt{r^2} \\ &= r \end{aligned}$$

$$\begin{aligned} h_\phi &= \left| \frac{\partial \mathbf{r}}{\partial \phi} \right| = \sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi} \\ &= \sqrt{r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi)} \\ &= \sqrt{r^2 \sin^2 \theta} \\ &= r \sin \theta \end{aligned}$$

### Differential Elements

#### Line Element

$$d\mathbf{r} = h_r dr \hat{\mathbf{r}} + h_\theta d\theta \hat{\boldsymbol{\theta}} + h_\phi d\phi \hat{\boldsymbol{\phi}} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}$$

#### Area Element

$$\begin{aligned} dA &= dA_{r=cte} = h_\theta h_\phi d\theta d\phi \hat{\mathbf{r}} = r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}} \\ &= dA_{\theta=cte} = h_r h_\phi dr d\phi \hat{\boldsymbol{\theta}} = r \sin \theta dr d\phi \hat{\boldsymbol{\theta}} \\ &= dA_{\phi=cte} = h_r h_\theta dr d\theta \hat{\boldsymbol{\phi}} = r dr d\theta \hat{\boldsymbol{\phi}} \end{aligned}$$

#### Volume Element

$$dV = h_r h_\theta h_\phi dr d\theta d\phi = r^2 \sin \theta dr d\theta d\phi$$

## Differential Operators

### Gradient

$$\begin{aligned}\nabla f &= \frac{1}{h_r} \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{h_\theta} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}\end{aligned}$$

### Divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{h_r h_\theta h_\phi} \left[ \frac{\partial(h_\theta h_\phi F_r)}{\partial r} + \frac{\partial(h_r h_\phi F_\theta)}{\partial \theta} + \frac{\partial(h_r h_\theta F_\phi)}{\partial \phi} \right] \\ &= \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial(r^2 F_r)}{\partial r} + r \frac{\partial(\sin \theta F_\theta)}{\partial \theta} + r \frac{\partial F_\phi}{\partial \phi} \right] \\ &= \frac{1}{r^2} \frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta F_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi}\end{aligned}$$

### Rotational

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{h_r h_\theta h_\phi} \begin{vmatrix} h_r \hat{\mathbf{r}} & h_\theta \hat{\boldsymbol{\theta}} & h_\phi \hat{\boldsymbol{\phi}} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ F_r h_r & F_\theta h_\theta & F_\phi h_\phi \end{vmatrix} \\ &= \frac{1}{h_r h_\theta h_\phi} \left[ h_r \left( \frac{\partial(h_\phi F_\phi)}{\partial \theta} - \frac{\partial(h_\theta F_\theta)}{\partial \phi} \right) \hat{\mathbf{r}} - h_\theta \left( \frac{\partial(h_\phi F_\phi)}{\partial r} - \frac{\partial(h_r F_r)}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + h_\phi \left( \frac{\partial(h_\theta F_\theta)}{\partial r} - \frac{\partial(h_r F_r)}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \right] \\ &= \frac{1}{r^2 \sin \theta} \left[ \left( r \frac{\partial(\sin \theta F_\phi)}{\partial \theta} - r \frac{\partial F_\theta}{\partial \phi} \right) \hat{\mathbf{r}} - r \left( \sin \theta \frac{\partial(r F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + r \sin \theta \left( \frac{\partial(r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \right] \\ &= \frac{1}{r^2 \sin \theta} \left( r \frac{\partial(\sin \theta F_\phi)}{\partial \theta} - r \frac{\partial F_\theta}{\partial \phi} \right) \hat{\mathbf{r}} - \frac{1}{r \sin \theta} \left( \sin \theta \frac{\partial(r F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left( \frac{\partial(r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \\ &= \frac{1}{r \sin \theta} \left( \frac{\partial(\sin \theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi} \right) \hat{\mathbf{r}} - \left( \frac{1}{r} \frac{\partial(r F_\phi)}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left( \frac{\partial(r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}}\end{aligned}$$

### Laplacian

$$\begin{aligned}\nabla^2 f &= \nabla \cdot (\nabla f) = \nabla \cdot \left( \frac{1}{h_r} \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{h_\theta} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} \right) \\ &= \frac{1}{h_r h_\theta h_\phi} \left[ \frac{\partial}{\partial r} \left( \frac{h_\theta h_\phi}{h_r} \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{h_r h_\phi}{h_\theta} \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{h_r h_\theta}{h_\phi} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}\end{aligned}$$

### 3 Cylindrical Polar Coordinates

$$(x, y, z) \longleftrightarrow (\rho, \phi, z)$$

$$x = r \cos \phi$$

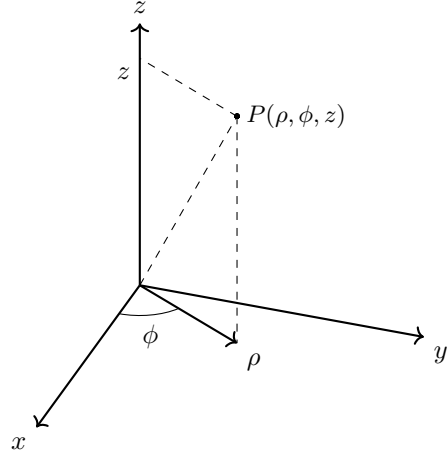
$$y = r \sin \phi$$

$$z = z$$

$$\rho \in [0, \infty), \quad \phi \in [0, 2\pi), \quad z \in (-\infty, \infty)$$

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$= r \cos \phi \hat{\boldsymbol{\rho}} + r \sin \phi \hat{\boldsymbol{\phi}} + z\hat{\mathbf{z}}$$



#### Scale Factors

$$h_\rho = \left| \frac{\partial \mathbf{r}}{\partial \rho} \right| = \sqrt{\cos^2 \phi + \sin^2 \phi} = 1$$

$$\begin{aligned} h_\phi &= \left| \frac{\partial \mathbf{r}}{\partial \phi} \right| = \sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi} \\ &= \sqrt{\rho^2 (\sin^2 \phi + \cos^2 \phi)} \\ &= \sqrt{\rho^2} \\ &= \rho \end{aligned}$$

$$h_z = \left| \frac{\partial \mathbf{r}}{\partial z} \right| = 1$$

#### Differential Elements

##### Line Element

$$d\mathbf{r} = h_\rho d\rho \hat{\boldsymbol{\rho}} + h_\phi d\phi \hat{\boldsymbol{\phi}} + h_z dz \hat{\mathbf{z}} = d\rho \hat{\boldsymbol{\rho}} + \rho d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$$

##### Area Element

$$\begin{aligned} dA &= dA_{\rho=cte} = h_\phi h_z d\phi dz \hat{\boldsymbol{\rho}} = \rho d\phi dz \hat{\boldsymbol{\rho}} \\ &= dA_{\phi=cte} = h_\rho h_z d\rho dz \hat{\boldsymbol{\phi}} = d\rho dz \hat{\boldsymbol{\phi}} \\ &= dA_{z=cte} = h_\rho h_\phi d\rho d\phi \hat{\mathbf{z}} = \rho d\rho d\phi \hat{\mathbf{z}} \end{aligned}$$

##### Volume Element

$$dV = h_\rho h_\phi h_z d\rho d\phi dz = \rho d\rho d\phi dz$$

#### Differential Operators

##### Gradient

$$\begin{aligned} \nabla f &= \frac{1}{h_\rho} \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{1}{h_z} \frac{\partial f}{\partial z} \hat{\mathbf{z}} \\ &= \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \end{aligned}$$

### Divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{h_\rho h_\phi h_z} \left[ \frac{\partial(h_\phi h_z F_\rho)}{\partial \rho} + \frac{\partial(h_\rho h_z F_\phi)}{\partial \phi} + \frac{\partial(h_\rho h_\phi F_z)}{\partial z} \right] \\ &= \frac{1}{\rho} \left[ \frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{\partial F_\phi}{\partial \phi} + \rho \frac{\partial F_z}{\partial z} \right] \\ &= \frac{1}{\rho} \frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}\end{aligned}$$

### Rotational

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{h_\rho h_\phi h_z} \begin{vmatrix} h_\rho \hat{\boldsymbol{\rho}} & h_\phi \hat{\boldsymbol{\phi}} & h_z \hat{\boldsymbol{z}} \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ F_\rho h_\rho & F_\phi h_\phi & F_z h_z \end{vmatrix} \\ &= \frac{1}{\rho} \left[ h_\rho \left( \frac{\partial(h_z F_z)}{\partial \phi} - \frac{\partial(h_\phi F_\phi)}{\partial z} \right) \hat{\boldsymbol{\rho}} - h_\phi \left( \frac{\partial(h_z F_z)}{\partial \rho} - \frac{\partial(h_\rho F_\rho)}{\partial z} \right) \hat{\boldsymbol{\phi}} + h_z \left( \frac{\partial(h_\phi F_\phi)}{\partial \rho} - \frac{\partial(h_\rho F_\rho)}{\partial \phi} \right) \hat{\boldsymbol{z}} \right] \\ &= \frac{1}{\rho} \left[ \left( \frac{\partial F_z}{\partial \phi} - \rho \frac{\partial F_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} - \rho \left( \frac{\partial F_z}{\partial \rho} - \frac{\partial F_\rho}{\partial z} \right) \hat{\boldsymbol{\phi}} + \left( \frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) \hat{\boldsymbol{z}} \right] \\ &= \left( \frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} - \left( \frac{\partial F_z}{\partial \rho} - \frac{\partial F_\rho}{\partial z} \right) \hat{\boldsymbol{\phi}} + \frac{1}{\rho} \left( \frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) \hat{\boldsymbol{z}}\end{aligned}$$

### Laplacian

$$\begin{aligned}\nabla^2 f &= \nabla \cdot (\nabla f) = \nabla \cdot \left( \frac{1}{h_\rho} \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{1}{h_z} \frac{\partial f}{\partial z} \hat{\boldsymbol{z}} \right) \\ &= \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left( \frac{\partial f}{\partial \phi} \right) + \rho \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial z} \right) \right] \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

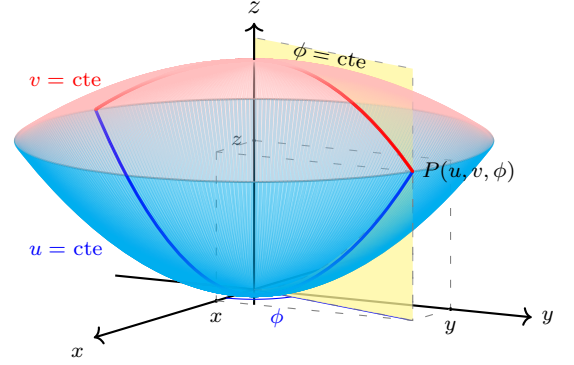
## 4 Parabolic Coordinates

$$(x, y, z) \longleftrightarrow (u, v, \phi)$$

$$\begin{aligned} x &= uv \cos \phi \\ y &= uv \sin \phi \\ z &= \frac{1}{2}(u^2 - v^2) \end{aligned}$$

$$u \in [0, \infty), \quad v \in [0, \infty), \quad \phi \in [0, 2\pi)$$

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\ &= uv \cos \phi \hat{\mathbf{u}} + uv \sin \phi \hat{\mathbf{v}} + \frac{1}{2}(u^2 - v^2)\hat{\phi} \end{aligned}$$



### Scale Factors

$$\begin{aligned} h_u &= \left| \frac{\partial \mathbf{r}}{\partial u} \right| = \sqrt{v^2 \cos^2 \phi + v^2 \sin^2 \phi + u^2} \\ &= \sqrt{v^2(\cos^2 \phi + \sin^2 \phi) + u^2} \\ &= \sqrt{v^2 + u^2} \end{aligned}$$

$$\begin{aligned} h_v &= \left| \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{u^2 \cos^2 \phi + u^2 \sin^2 \phi + v^2} \\ &= \sqrt{u^2(\cos^2 \phi + \sin^2 \phi) + v^2} \\ &= \sqrt{u^2 + v^2} \end{aligned}$$

$$\begin{aligned} h_\phi &= \left| \frac{\partial \mathbf{r}}{\partial \phi} \right| = \sqrt{u^2 v^2 \sin^2 \phi + u^2 v^2 \cos^2 \phi} \\ &= \sqrt{u^2 v^2(\sin^2 \phi + \cos^2 \phi)} \\ &= \sqrt{u^2 v^2} \\ &= uv \end{aligned}$$

### Differential Elements

#### Line Element

$$du = h_u du \hat{\mathbf{u}} + h_\phi + h_v dv \hat{\mathbf{v}} + d\phi \hat{\phi} = \sqrt{v^2 + u^2} du \hat{\mathbf{u}} + \sqrt{u^2 + v^2} dv \hat{\mathbf{v}} + uv d\phi \hat{\phi}$$

#### Area Element

$$\begin{aligned} dA &= dA_{u=cte} = h_v h_\phi dv d\phi \hat{\mathbf{u}} = uv \sqrt{u^2 + v^2} dv d\phi \hat{\mathbf{u}} \\ &= dA_{v=cte} = h_u h_\phi du d\phi \hat{\mathbf{v}} = uv \sqrt{v^2 + u^2} du d\phi \hat{\mathbf{v}} \\ &= dA_{\phi=cte} = h_u h_v du dv \hat{\phi} = (u^2 + v^2) du dv \hat{\phi} \end{aligned}$$

#### Volume Element

$$dV = h_u h_v h_\phi du dv d\phi = uv(u^2 + v^2) du dv d\phi$$

## Differential Operators

### Gradient

$$\begin{aligned}\nabla f &= \frac{1}{h_u} \frac{\partial f}{\partial u} \hat{\mathbf{u}} + \frac{1}{h_v} \frac{\partial f}{\partial v} \hat{\mathbf{v}} + \frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial f}{\partial u} \hat{\mathbf{u}} + \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial f}{\partial v} \hat{\mathbf{v}} + \frac{1}{uv} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}\end{aligned}$$

### Divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{h_u h_v h_\phi} \left[ \frac{\partial(h_v h_\phi F_u)}{\partial u} + \frac{\partial(h_u h_\phi F_v)}{\partial v} + \frac{\partial(h_u h_v F_\phi)}{\partial \phi} \right] \\ &= \frac{1}{uv(u^2 + v^2)} \left[ v \frac{\partial(u\sqrt{u^2 + v^2} F_u)}{\partial u} + u \frac{\partial(v\sqrt{u^2 + v^2} F_v)}{\partial v} + (u^2 + v^2) \frac{\partial F_\phi}{\partial \phi} \right] \\ &= \frac{1}{u(u^2 + v^2)} \frac{\partial(u\sqrt{u^2 + v^2} F_u)}{\partial u} + \frac{1}{v(u^2 + v^2)} \frac{\partial(v\sqrt{u^2 + v^2} F_v)}{\partial v} + \frac{1}{uv} \frac{\partial F_\phi}{\partial \phi}\end{aligned}$$

### Rotational

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{h_u h_v h_\phi} \begin{vmatrix} h_u \hat{\mathbf{u}} & h_v \hat{\mathbf{v}} & h_\phi \hat{\boldsymbol{\phi}} \\ \partial/\partial u & \partial/\partial v & \partial/\partial \phi \\ F_u h_u & F_v h_v & F_\phi h_\phi \end{vmatrix} \\ &= \frac{1}{h_u h_v h_\phi} \left[ h_u \left( \frac{\partial(h_\phi F_\phi)}{\partial v} - \frac{\partial(h_v F_v)}{\partial \phi} \right) \hat{\mathbf{u}} - h_v \left( \frac{\partial(h_\phi F_\phi)}{\partial u} - \frac{\partial(h_u F_u)}{\partial \phi} \right) \hat{\mathbf{v}} + h_\phi \left( \frac{\partial(h_v F_v)}{\partial u} - \frac{\partial(h_u F_u)}{\partial v} \right) \hat{\boldsymbol{\phi}} \right] \\ &= \frac{1}{uv(u^2 + v^2)} \left[ \sqrt{u^2 + v^2} \left( u \frac{\partial(v F_\phi)}{\partial v} - \sqrt{u^2 + v^2} \frac{\partial F_v}{\partial \phi} \right) \hat{\mathbf{u}} - \sqrt{u^2 + v^2} \left( v \frac{\partial(u F_\phi)}{\partial u} - (u^2 + v^2) \frac{\partial F_u}{\partial \phi} \right) \hat{\mathbf{v}} \right. \\ &\quad \left. + \left( uv \frac{\partial(\sqrt{u^2 + v^2} F_v)}{\partial u} - uv \frac{\partial(\sqrt{u^2 + v^2} F_u)}{\partial v} \right) \hat{\boldsymbol{\phi}} \right] \\ &= \frac{1}{v} \left( \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial(v F_\phi)}{\partial v} - \frac{1}{u} \frac{\partial F_v}{\partial \phi} \right) \hat{\mathbf{u}} - \frac{1}{u} \left( \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial(u F_\phi)}{\partial u} - \frac{1}{v} \frac{\partial F_u}{\partial \phi} \right) \hat{\mathbf{v}} \\ &\quad + \frac{1}{(u^2 + v^2)} \left( \frac{\partial(\sqrt{u^2 + v^2} F_v)}{\partial u} - \frac{\partial(\sqrt{u^2 + v^2} F_u)}{\partial v} \right) \hat{\boldsymbol{\phi}} \\ &= \frac{1}{r \sin \theta} \left( \frac{\partial(\sin \theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi} \right) \hat{\mathbf{r}} - \left( \frac{1}{r} \frac{\partial(r F_\phi)}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left( \frac{\partial(r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}}\end{aligned}$$

## Laplacian

$$\begin{aligned}\nabla^2 f &= \nabla \cdot (\nabla f) = \nabla \cdot \left( \frac{1}{h_u} \frac{\partial f}{\partial u} \hat{\mathbf{u}} + \frac{1}{h_v} \frac{\partial f}{\partial v} \hat{\mathbf{v}} + \frac{1}{h_\phi} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} \right) \\ &= \frac{1}{h_u h_v h_\phi} \left[ \frac{\partial}{\partial u} \left( \frac{h_v h_\phi}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{h_u h_\phi}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial \phi} \left( \frac{h_u h_v}{h_\phi} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{uv(u^2 + v^2)} \left[ v \frac{\partial}{\partial u} \left( u \frac{\partial f}{\partial u} \right) + u \frac{\partial}{\partial v} \left( v \frac{\partial f}{\partial v} \right) + \frac{(u^2 + v^2)}{uv} \frac{\partial}{\partial \phi} \left( \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{u(u^2 + v^2)} \frac{\partial}{\partial u} \left( u \frac{\partial f}{\partial u} \right) + \frac{1}{v(u^2 + v^2)} \frac{\partial}{\partial v} \left( v \frac{\partial f}{\partial v} \right) + \frac{1}{u^2 v^2} \frac{\partial^2 f}{\partial \phi^2}\end{aligned}$$