Linear Momentum Conservation

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First, let's see some definitions:

$$\boxed{\sum_{i,j} \mathbf{F}_{ij} = \sum_{i,j} \mathbf{F}_{ji} = \sum_{j,i} \mathbf{F}_{ij} = \sum_{j,i} \mathbf{F}_{ji}} \tag{1}$$

$$\begin{aligned}
\sum_{i,j}^{N} \mathbf{F}_{ij} &= \sum_{j,i}^{N} \mathbf{F}_{ji} \\
\sum_{i,j}^{N} \mathbf{F}_{ij} + \sum_{i,j}^{N} \mathbf{F}_{ij} &= \sum_{i,j}^{N} \mathbf{F}_{ij} + \sum_{j,i}^{N} \mathbf{F}_{ji} \\
2\sum_{i,j}^{N} \mathbf{F}_{ij} &= \sum_{i,j}^{N} \mathbf{F}_{ij} + \sum_{j,i}^{N} \mathbf{F}_{ji} \\
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\end{aligned}$$

$$\boxed{\sum_{i,j}^{N} \mathbf{F}_{ij} = \frac{1}{2}\sum_{i,j}^{N} (\mathbf{F}_{ij} + \mathbf{F}_{ji})} \tag{2}$$

Now, we start from the Newton's second law:

$$\mathbf{F} = m\mathbf{a}. \tag{3}$$

For a system with several particles, it is assumed that the force on each particle can be decomposed in **externeal forces** (exerted by sources outside the system) and **internal forces** (exerted by the other particles of the system). Then, for the ith particle of the system of N particles, equation (3) turns out

$$\dot{\mathbf{p}}_i = \sum_{\substack{i,j \\ i\neq j}}^{N} \mathbf{F}_{ij} + \mathbf{F}_i^{(e)}, \tag{4}$$

where

$$\mathbf{p}_i = m_i \mathbf{v}_i = m_i \dot{\mathbf{r}}_i \tag{5}$$

is the **linear momentum** of the ith particle, m_i its mass, $\mathbf{r}_i$ its position vector, $\mathbf{v}_i$ its velocity and $\mathbf{F}_i^{(e)}$ is the net external force on it.

$$\sum_{i}^{N} m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{\substack{i,j \\ i\neq j}}^{N} \mathbf{F}_{ij} + \mathbf{F}_i^{(e)} = \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}}^{N} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) + \mathbf{F}_i^{(e)}. \tag{6}$$

From the Newton's third law we know that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \tag{7}$$

therefore, the first term in (6) vanishes.

$$\dot{\mathbf{p}}_i = \frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i^{(e)}. \tag{8}$$

Defining the center-of-mass position vector by

$$\mathbf{R} = \frac{\sum_i^N m_i \mathbf{r}_i}{\sum_i^N m_i} = \frac{\sum_i^N m_i \mathbf{r}_i}{M}, \tag{9}$$

the equation (8) can be written as

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{(e)}. \tag{10}$$

If there is no external force, that means $\mathbf{F}^{(e)} = 0$. So, integrating both sides we obtain

$$\boxed{\mathbf{P} = constant} \tag{11}$$

which means that the total linear momentum of a system of particles is conserved.