Schwarzschild Solution Derivation

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1 Prelude

We use the metric with signature $(-+++)$. We assume cosmological constant $\Lambda = 0$. Unlike other derivations, in this intuitive work I want to make clear some points and convince the reader that every step I do makes sense without so much mathematical rigor.

2 Introduction

The Einstein Field Equation (EFE) is

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

$$R_{\mu\nu} = \text{Ricci tensor}, \quad R = \text{Ricci scalar}, \quad T_{\mu\nu} = \text{Energy-Momentum tensor}$$

We want to describe the exterior of a black hole, this means $T_{\mu\nu} = 0$. Solving EFE gives us

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 0 \qquad / \cdot g^{\mu\nu} \\
R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} &= 0 \qquad \implies R_{\mu\nu}g^{\mu\nu} = R^{\mu}_{\mu} = R \\
R^{\mu}_{\mu} - \frac{1}{2}R\delta^{\mu}_{\mu} &= 0 \\
R - \frac{1}{2}R(4) &= 0 \\
R - 2R &= 0 \\
R &= 0 \qquad \implies R_{\mu\nu} = 0 \quad \text{"Ricci plane"}
\end{aligned}$$

Ricci plane doesn't mean plane space-time. It means area or volume doesn't change along the geodesic.

The Minkowski metric in spherical coordinates (r, θ, ϕ) is

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

The metric tensor $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{tt} & g_{tr} & g_{t\theta} & g_{t\phi} \\ g_{rt} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta t} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi t} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$

3 Assumptions

1. **The system is spherically symmetric**

   This means we can use spherical coordinates for θ and ϕ.

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tr} & g_{t\theta} & g_{t\phi} \\ g_{rt} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta t} & g_{\theta r} & r^2 & 0 \\ g_{\phi t} & g_{\phi r} & 0 & r^2 \sin^2\theta \end{pmatrix}$$

   Also this means we can introduce functions without breaking the symmetry (we will see later).

2. **The system is static**

   Static means two things:

   - Stationary space-time
   - System invariant under $t \to -t$ transformation

   This means that the system is independence of time and is not rotating, which also means independence of θ and ϕ.

We can introduce two functions $g_{tt} = A(r)$ and $g_{rr} = B(r)$.

The metric tensor $g_{\mu\nu}$ turns out

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

In order to obtain $A(r)$ and $B(r)$ we need to:

- Calculate Christoffel symbols (connection coefficients).
- Calculate Ricci tensor components.
- Match the results with the Newtonian gravity theory at some boundaries.

To simplify calculations we set

$$A(r) = A \qquad B(r) = B$$

$$\frac{d}{dr}A(r) = A' \qquad \frac{d}{dr}B(r) = B'(r)$$

4 Christoffel Symbols

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})}$$

Since $g_{\mu\nu}$ is diagonal we know that

$$g_{00} = -A \qquad g_{11} = B \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin\theta$$

$$g^{00} = -\frac{1}{A} \qquad g^{11} = \frac{1}{B} \qquad g^{22} = \frac{1}{r^2} \qquad g^{33} = \frac{1}{r^2 \sin\theta}$$

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2} g^{0\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) \\
&= \frac{1}{2} g^{00}(\cancelto{0}{\partial_0 g_{00}} + \cancelto{0}{\partial_0 g_{00}} - \cancelto{0}{\partial_0 g_{00}}) + \frac{1}{2}\cancelto{0}{g^{01}}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}) \\
&+ \frac{1}{2}\cancelto{0}{g^{02}}(\partial_0 g_{20} + \partial_0 g_{20} - \partial_2 g_{00}) + \frac{1}{2}\cancelto{0}{g^{03}}(\partial_0 g_{30} + \partial_0 g_{30} - \partial_3 g_{00}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2} g^{0\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) \\
&= \frac{1}{2} g^{00}(\partial_1 \cancelto{0}{g_{01}} + \partial_1 \cancelto{0}{g_{01}} - \cancelto{0}{\partial_0 g_{11}}) + \frac{1}{2}\cancelto{0}{g^{01}}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\
&+ \frac{1}{2}\cancelto{0}{g^{02}}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) + \frac{1}{2}\cancelto{0}{g^{03}}(\partial_1 g_{31} + \partial_1 g_{31} - \partial_3 g_{11}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{22} &= \frac{1}{2} g^{0\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) \\
&= \frac{1}{2} g^{00}(\partial_2 \cancelto{0}{g_{02}} + \partial_2 \cancelto{0}{g_{02}} - \cancelto{0}{\partial_0 g_{22}}) + \frac{1}{2}\cancelto{0}{g^{01}}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) \\
&+ \frac{1}{2}\cancelto{0}{g^{02}}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) + \frac{1}{2}\cancelto{0}{g^{03}}(\partial_2 g_{32} + \partial_2 g_{32} - \partial_3 g_{22}) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\Gamma^0_{33} &= \frac{1}{2} g^{3\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33}) \\
&= \frac{1}{2} g^{00}(\partial_3 \cancelto{0}{g_{03}} + \partial_3 \cancelto{0}{g_{03}} - \cancelto{0}{\partial_0} g_{33}) + \frac{1}{2} \cancelto{0}{g^{01}}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) \\
&+ \frac{1}{2} \cancelto{0}{g^{02}}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) + \frac{1}{2} \cancelto{0}{g^{03}}(\partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{10} = \Gamma^0_{01} &= \frac{1}{2} g^{0\alpha}(\partial_0 g_{\alpha 1} + \partial_1 g_{\alpha 0} - \partial_\alpha g_{01}) \\
&= \frac{1}{2} g^{00}(\cancel{\partial_0 g_{01}} + \partial_1 g_{00} - \cancel{\partial_0 g_{01}}) + \frac{1}{2} \cancelto{0}{g^{01}}(\partial_0 g_{11} + \partial_1 g_{10} - \partial_1 g_{01}) \\
&+ \frac{1}{2} \cancelto{0}{g^{02}}(\partial_0 g_{21} + \partial_1 g_{20} - \partial_2 g_{01}) + \frac{1}{2} \cancelto{0}{g^{03}}(\partial_0 g_{31} + \partial_1 g_{30} - \partial_3 g_{01}) \\
&= \frac{1}{2}\left(-\frac{1}{A}\right)\left(-A'\right) = \frac{1}{2}\frac{A'}{A}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{20} = \Gamma^0_{02} &= \frac{1}{2} g^{0\alpha}(\partial_0 g_{\alpha 2} + \partial_2 g_{\alpha 0} - \partial_\alpha g_{02}) \\
&= \frac{1}{2} g^{00}(\cancel{\partial_0 g_{02}} + \cancelto{0}{\partial_2} g_{00} - \cancel{\partial_0 g_{02}}) + \frac{1}{2} \cancelto{0}{g^{01}}(\partial_0 g_{12} + \partial_2 g_{10} - \partial_1 g_{02}) \\
&+ \frac{1}{2} \cancelto{0}{g^{02}}(\partial_0 g_{22} + \partial_2 g_{20} - \partial_2 g_{02}) + \frac{1}{2} \cancelto{0}{g^{03}}(\partial_0 g_{32} + \partial_2 g_{30} - \partial_3 g_{02}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{30} = \Gamma^0_{03} &= \frac{1}{2} g^{0\alpha}(\partial_0 g_{\alpha 3} + \partial_3 g_{\alpha 0} - \partial_\alpha g_{03}) \\
&= \frac{1}{2} g^{00}(\cancel{\partial_0 g_{03}} + \cancelto{0}{\partial_3} g_{00} - \cancel{\partial_0 g_{03}}) + \frac{1}{2} \cancelto{0}{g^{01}}(\partial_0 g_{13} + \partial_3 g_{10} - \partial_1 g_{03}) \\
&+ \frac{1}{2} \cancelto{0}{g^{02}}(\partial_0 g_{23} + \partial_3 g_{20} - \partial_2 g_{03}) + \frac{1}{2} \cancelto{0}{g^{03}}(\partial_0 g_{33} + \partial_3 g_{30} - \partial_3 g_{03}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{21} = \Gamma^0_{12} &= \frac{1}{2}g^{0\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) \\
&= \frac{1}{2}g^{00}(\partial_1 \cancelto{0}{g_{02}} + \partial_2 \cancelto{0}{g_{01}} - \partial_0 \cancelto{0}{g_{12}}) + \frac{1}{2}\cancelto{0}{g^{01}}(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}) \\
&+ \frac{1}{2}\cancelto{0}{g^{02}}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{12}) + \frac{1}{2}\cancelto{0}{g^{03}}(\partial_1 g_{32} + \partial_2 g_{31} - \partial_3 g_{12}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{31} = \Gamma^0_{13} &= \frac{1}{2}g^{0\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) \\
&= \frac{1}{2}g^{00}(\partial_1 \cancelto{0}{g_{03}} + \partial_3 \cancelto{0}{g_{01}} - \partial_0 \cancelto{0}{g_{13}}) + \frac{1}{2}\cancelto{0}{g^{01}}(\partial_1 g_{13} + \partial_3 g_{11} - \partial_1 g_{13}) \\
&+ \frac{1}{2}\cancelto{0}{g^{02}}(\partial_1 g_{23} + \partial_3 g_{21} - \partial_2 g_{13}) + \frac{1}{2}\cancelto{0}{g^{03}}(\partial_1 g_{33} + \partial_3 g_{31} - \partial_3 g_{13}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^0_{32} = \Gamma^0_{23} &= \frac{1}{2}g^{0\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23}) \\
&= \frac{1}{2}g^{00}(\partial_2 \cancelto{0}{g_{03}} + \partial_3 \cancelto{0}{g_{02}} - \partial_0 \cancelto{0}{g_{23}}) + \frac{1}{2}g^{01}(\partial_2 g_{13} + \partial_3 g_{12} - \partial_1 g_{23}) \\
&+ \frac{1}{2}\cancelto{0}{g^{02}}(\partial_2 g_{23} + \partial_3 g_{22} - \partial_2 g_{23}) + \frac{1}{2}\cancelto{0}{g^{03}}(\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{1\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) \\
&= \frac{1}{2}\cancelto{0}{g^{10}}(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) + \frac{1}{2}g^{11}(\partial_0 \cancelto{0}{g_{10}} + \partial_0 \cancelto{0}{g_{10}} - \partial_1 g_{00}) \\
&+ \frac{1}{2}\cancelto{0}{g^{12}}(\partial_0 g_{20} + \partial_0 g_{20} - \partial_2 g_{00}) + \frac{1}{2}\cancelto{0}{g^{13}}(\partial_0 g_{30} + \partial_0 g_{30} - \partial_3 g_{00}) \\
&= -\frac{1}{2}\left(\frac{1}{B}\right)(-A') = \frac{1}{2}\frac{A}{B}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{1\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) \\
&= \frac{1}{2}\cancelto{0}{g^{10}}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) + \frac{1}{2} g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\
&+ \frac{1}{2}\cancelto{0}{g^{12}}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) + \frac{1}{2}\cancelto{0}{g^{13}}(\partial_1 g_{31} + \partial_1 g_{31} - \partial_3 g_{11}) \\
&= \frac{1}{2}\left(\frac{1}{B}\right)(B') = \frac{1}{2}\frac{B'}{B}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2} g^{1\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) \\
&= \frac{1}{2}\cancelto{0}{g^{10}}(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22}) + \frac{1}{2} g^{11}(\partial_2 \cancelto{0}{g_{12}} + \partial_2 \cancelto{0}{g_{12}} - \partial_1 g_{22}) \\
&+ \frac{1}{2}\cancelto{0}{g^{12}}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) + \frac{1}{2}\cancelto{0}{g^{13}}(\partial_2 g_{32} + \partial_2 g_{32} - \partial_3 g_{22}) \\
&= -\frac{1}{2}\left(\frac{1}{B}\right)(2r) = -\frac{r}{B}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2} g^{1\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33}) \\
&= \frac{1}{2}\cancelto{0}{g^{10}}(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}) + \frac{1}{2} g^{11}(\partial_3 \cancelto{0}{g_{13}} + \partial_3 \cancelto{0}{g_{13}} - \partial_1 g_{33}) \\
&+ \frac{1}{2}\cancelto{0}{g^{12}}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) + \frac{1}{2}\cancelto{0}{g^{13}}(\partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33}) \\
&= -\frac{1}{2}\left(\frac{1}{B}\right)\left(2r\sin^2\theta\right) = -\frac{r\sin^2\theta}{B}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{10} = \Gamma^1_{01} &= \frac{1}{2} g^{1\alpha}(\partial_0 g_{\alpha 1} + \partial_1 g_{\alpha 0} - \partial_\alpha g_{01}) \\
&= \frac{1}{2}\cancelto{0}{g^{10}}(\partial_0 g_{01} + \partial_1 g_{00} - \partial_0 g_{01}) + \frac{1}{2} g^{11}(\cancelto{0}{\partial_0} g_{11} + \cancel{\partial_1 g_{10}} - \cancel{\partial_1 g_{01}}) \\
&+ \frac{1}{2}\cancelto{0}{g^{12}}(\partial_0 g_{21} + \partial_1 g_{20} - \partial_2 g_{01}) + \frac{1}{2}\cancelto{0}{g^{13}}(\partial_0 g_{31} + \partial_1 g_{30} - \partial_3 g_{01}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{20} = \Gamma^1_{02} &= \frac{1}{2} g^{1\alpha}(\partial_0 g_{\alpha 2} + \partial_2 g_{\alpha 0} - \partial_\alpha g_{02}) \\
&= \frac{1}{2} \cancelto{0}{g^{10}}(\partial_0 g_{02} + \partial_2 g_{00} - \partial_0 g_{02}) + \frac{1}{2} g^{11}(\partial_0 \cancelto{0}{g_{12}} + \partial_2 \cancelto{0}{g_{10}} - \partial_1 \cancelto{0}{g_{02}}) \\
&+ \frac{1}{2} \cancelto{0}{g^{12}}(\partial_0 g_{22} + \partial_2 g_{20} - \partial_2 g_{02}) + \frac{1}{2} \cancelto{0}{g^{13}}(\partial_0 g_{32} + \partial_2 g_{30} - \partial_3 g_{02}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{30} = \Gamma^1_{03} &= \frac{1}{2} g^{1\alpha}(\partial_0 g_{\alpha 3} + \partial_3 g_{\alpha 0} - \partial_\alpha g_{03}) \\
&= \frac{1}{2} \cancelto{0}{g^{10}}(\partial_0 g_{03} + \partial_3 g_{00} - \partial_0 g_{03}) + \frac{1}{2} g^{11}(\partial_0 \cancelto{0}{g_{13}} + \partial_3 \cancelto{0}{g_{10}} - \partial_1 \cancelto{0}{g_{03}}) \\
&+ \frac{1}{2} \cancelto{0}{g^{12}}(\partial_0 g_{23} + \partial_3 g_{20} - \partial_2 g_{03}) + \frac{1}{2} \cancelto{0}{g^{13}}(\partial_0 g_{33} + \partial_3 g_{30} - \partial_3 g_{03}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{21} = \Gamma^1_{12} &= \frac{1}{2} g^{1\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) \\
&= \frac{1}{2} \cancelto{0}{g^{10}}(\partial_1 g_{02} + \partial_2 g_{01} - \partial_0 g_{12}) + \frac{1}{2} g^{11}(\cancel{\partial_1 g_{12}} + \cancelto{0}{\partial_2} g_{11} - \cancel{\partial_1 g_{12}}) \\
&+ \frac{1}{2} \cancelto{0}{g^{12}}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{12}) + \frac{1}{2} \cancelto{0}{g^{13}}(\partial_1 g_{32} + \partial_2 g_{31} - \partial_3 g_{12}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{31} = \Gamma^1_{13} &= \frac{1}{2} g^{1\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) \\
&= \frac{1}{2} \cancelto{0}{g^{10}}(\partial_1 g_{03} + \partial_3 g_{01} - \partial_0 g_{13}) + \frac{1}{2} g^{11}(\cancel{\partial_1 g_{13}} + \cancelto{0}{\partial_3} g_{11} - \cancel{\partial_1 g_{13}}) \\
&+ \frac{1}{2} \cancelto{0}{g^{12}}(\partial_1 g_{23} + \partial_3 g_{21} - \partial_2 g_{13}) + \frac{1}{2} \cancelto{0}{g^{13}}(\partial_1 g_{33} + \partial_3 g_{31} - \partial_3 g_{13}) \\
&= 0
\end{aligned}
$$

$$\Gamma^1_{32} = \Gamma^1_{23} = \frac{1}{2} g^{1\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23})$$

$$= \frac{1}{2} \cancelto{0}{g^{10}}(\partial_2 g_{03} + \partial_3 g_{02} - \partial_0 g_{23}) + \frac{1}{2} g^{11}(\partial_2 \cancelto{0}{g_{13}} + \partial_3 \cancelto{0}{g_{12}} - \partial_1 \cancelto{0}{g_{23}})$$

$$+ \frac{1}{2} \cancelto{0}{g^{12}}(\partial_2 g_{23} + \partial_3 g_{22} - \partial_2 g_{23}) + \frac{1}{2} \cancelto{0}{g^{13}}(\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23})$$

$$= 0$$

$$\Gamma^2_{00} = \frac{1}{2} g^{2\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00})$$

$$= \frac{1}{2} \cancelto{0}{g^{20}}(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) + \frac{1}{2} \cancelto{0}{g^{21}}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00})$$

$$+ \frac{1}{2} g^{22}(\partial_0 \cancelto{0}{g_{20}} + \partial_0 \cancelto{0}{g_{20}} - \cancelto{0}{\partial_2} g_{00}) + \frac{1}{2} \cancelto{0}{g^{23}}(\partial_0 g_{30} + \partial_0 g_{30} - \partial_3 g_{00})$$

$$= 0$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11})$$

$$= \frac{1}{2} \cancelto{0}{g^{20}}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) + \frac{1}{2} \cancelto{0}{g^{21}}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11})$$

$$+ \frac{1}{2} g^{22}(\partial_1 \cancelto{0}{g_{21}} + \partial_1 \cancelto{0}{g_{21}} - \cancelto{0}{\partial_2} g_{11}) + \frac{1}{2} \cancelto{0}{g^{23}}(\partial_1 g_{31} + \partial_1 g_{31} - \partial_3 g_{11})$$

$$= 0$$

$$\Gamma^2_{22} = \frac{1}{2} g^{2\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22})$$

$$= \frac{1}{2} \cancelto{0}{g^{20}}(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22}) + \frac{1}{2} \cancelto{0}{g^{21}}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22})$$

$$+ \frac{1}{2} g^{22}(\cancelto{0}{\partial_2} g_{22} + \cancel{\partial_2 g_{22}} - \cancel{\partial_2 g_{22}}) + \frac{1}{2} \cancelto{0}{g^{23}}(\partial_2 g_{32} + \partial_2 g_{32} - \partial_3 g_{22})$$

$$= 0$$

$$\Gamma^2_{33} = \frac{1}{2}g^{2\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33})$$

$$= \frac{1}{2}\cancelto{0}{g^{20}}(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}) + \frac{1}{2}\cancelto{0}{g^{21}}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33})$$

$$+ \frac{1}{2}g^{22}(\partial_3 \cancelto{0}{g_{23}} + \partial_3 \cancelto{0}{g_{23}} - \partial_2 g_{33}) + \frac{1}{2}\cancelto{0}{g^{23}}(\partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33})$$

$$= -\frac{1}{2}\left(\frac{1}{r^2}\right)\left(2r^2 \sin\theta\cos\theta\right) = -\sin\theta\cos\theta$$

$$\Gamma^2_{10} = \Gamma^2_{01} = \frac{1}{2}g^{2\alpha}(\partial_0 g_{\alpha 1} + \partial_1 g_{\alpha 0} - \partial_\alpha g_{01})$$

$$= \frac{1}{2}\cancelto{0}{g^{20}}(\partial_0 g_{01} + \partial_1 g_{00} - \partial_0 g_{01}) + \frac{1}{2}\cancelto{0}{g^{21}}(\partial_0 g_{11} + \partial_1 g_{10} - \partial_1 g_{01})$$

$$+ \frac{1}{2}g^{22}(\partial_0 \cancelto{0}{g_{21}} + \partial_1 \cancelto{0}{g_{20}} - \partial_2 \cancelto{0}{g_{01}}) + \frac{1}{2}\cancelto{0}{g^{23}}(\partial_0 g_{31} + \partial_1 g_{30} - \partial_3 g_{01})$$

$$= 0$$

$$\Gamma^2_{20} = \Gamma^2_{02} = \frac{1}{2}g^{2\alpha}(\partial_0 g_{\alpha 2} + \partial_2 g_{\alpha 0} - \partial_\alpha g_{02})$$

$$= \frac{1}{2}\cancelto{0}{g^{20}}(\partial_0 g_{02} + \partial_2 g_{00} - \partial_0 g_{02}) + \frac{1}{2}\cancelto{0}{g^{21}}(\partial_0 g_{12} + \partial_2 g_{10} - \partial_1 g_{02})$$

$$+ \frac{1}{2}g^{22}(\cancelto{0}{\partial_0} g_{22} + \partial_2 \cancelto{0}{g_{20}} - \partial_2 \cancelto{0}{g_{02}}) + \frac{1}{2}\cancelto{0}{g^{23}}(\partial_0 g_{32} + \partial_2 g_{30} - \partial_3 g_{02})$$

$$= 0$$

$$\Gamma^2_{30} = \Gamma^2_{03} = \frac{1}{2}g^{2\alpha}(\partial_0 g_{\alpha 3} + \partial_3 g_{\alpha 0} - \partial_\alpha g_{03})$$

$$= \frac{1}{2}\cancelto{0}{g^{20}}(\partial_0 g_{03} + \partial_3 g_{00} - \partial_0 g_{03}) + \frac{1}{2}\cancelto{0}{g^{21}}(\partial_0 g_{13} + \partial_3 g_{10} - \partial_1 g_{03})$$

$$+ \frac{1}{2}g^{22}(\partial_0 \cancelto{0}{g_{23}} + \partial_3 \cancelto{0}{g_{20}} - \partial_2 \cancelto{0}{g_{03}}) + \frac{1}{2}\cancelto{0}{g^{23}}(\partial_0 g_{33} + \partial_3 g_{30} - \partial_3 g_{03})$$

$$= 0$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2} g^{2\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12})$$

$$= \frac{1}{2} \cancelto{0}{g^{20}}(\partial_1 g_{02} + \partial_2 g_{01} - \partial_0 g_{12}) + \frac{1}{2} \cancelto{0}{g^{21}}(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12})$$

$$+ \frac{1}{2} g^{22}(\partial_1 g_{22} + \partial_2 \cancelto{0}{g_{21}} - \partial_2 \cancelto{0}{g_{12}}) + \frac{1}{2} \cancelto{0}{g^{23}}(\partial_1 g_{32} + \partial_2 g_{31} - \partial_3 g_{12})$$

$$= \frac{1}{2} \left(\frac{1}{r^2 \sin^2 \theta} \right) \left(2r \sin^2 \theta \right) = \frac{1}{r}$$

$$\Gamma^2_{31} = \Gamma^2_{13} = \frac{1}{2} g^{2\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13})$$

$$= \frac{1}{2} \cancelto{0}{g^{20}}(\partial_1 g_{03} + \partial_3 g_{01} - \partial_0 g_{13}) + \frac{1}{2} \cancelto{0}{g^{21}}(\partial_1 g_{13} + \partial_3 g_{11} - \partial_1 g_{13})$$

$$+ \frac{1}{2} g^{22}(\partial_1 \cancelto{0}{g_{23}} + \partial_3 \cancelto{0}{g_{21}} - \partial_2 \cancelto{0}{g_{13}}) + \frac{1}{2} \cancelto{0}{g^{23}}(\partial_1 g_{33} + \partial_3 g_{31} - \partial_3 g_{13})$$

$$= 0$$

$$\Gamma^2_{32} = \Gamma^2_{23} = \frac{1}{2} g^{2\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23})$$

$$= \frac{1}{2} \cancelto{0}{g^{20}}(\partial_2 g_{03} + \partial_3 g_{02} - \partial_0 g_{23}) + \frac{1}{2} \cancelto{0}{g^{21}}(\partial_2 g_{13} + \partial_3 g_{12} - \partial_1 g_{23})$$

$$+ \frac{1}{2} g^{22}(\partial_2 \cancelto{0}{g_{23}} + \partial_3 \cancelto{0}{g_{22}} - \partial_2 \cancelto{0}{g_{23}}) + \frac{1}{2} \cancelto{0}{g^{23}}(\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23})$$

$$= 0$$

$$\Gamma^3_{00} = \frac{1}{2} g^{3\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00})$$

$$= \frac{1}{2} \cancelto{0}{g^{30}}(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) + \frac{1}{2} \cancelto{0}{g^{31}}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00})$$

$$+ \frac{1}{2} \cancelto{0}{g^{32}}(\partial_0 g_{20} + \partial_0 g_{20} - \partial_2 g_{00}) + \frac{1}{2} g^{33}(\partial_0 \cancelto{0}{g_{30}} + \partial_0 \cancelto{0}{g_{30}} - \cancelto{0}{\partial_3 g_{00}})$$

$$= 0$$

$$
\begin{aligned}
\Gamma^3_{11} &= \frac{1}{2} g^{3\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) \\
&= \frac{1}{2} \cancelto{0}{g^{30}}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) + \frac{1}{2} \cancelto{0}{g^{31}}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\
&+ \frac{1}{2} \cancelto{0}{g^{32}}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) + \frac{1}{2} g^{33}(\partial_1 \cancelto{0}{g_{31}} + \partial_1 \cancelto{0}{g_{31}} - \cancelto{0}{\partial_3} g_{11}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{22} &= \frac{1}{2} g^{3\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) \\
&= \frac{1}{2} \cancelto{0}{g^{30}}(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22}) + \frac{1}{2} \cancelto{0}{g^{31}}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) \\
&+ \frac{1}{2} \cancelto{0}{g^{32}}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) + \frac{1}{2} g^{33}(\partial_2 \cancelto{0}{g_{32}} + \partial_2 \cancelto{0}{g_{32}} - \cancelto{0}{\partial_3} g_{22}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{33} &= \frac{1}{2} g^{3\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33}) \\
&= \frac{1}{2} \cancelto{0}{g^{30}}(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}) + \frac{1}{2} \cancelto{0}{g^{31}}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) \\
&+ \frac{1}{2} \cancelto{0}{g^{32}}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) + \frac{1}{2} g^{33}(\cancelto{0}{\partial_3} g_{33} + \partial_3 g_{33} - \partial_3 g_{33}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{10} = \Gamma^3_{01} &= \frac{1}{2} g^{3\alpha}(\partial_0 g_{\alpha 1} + \partial_1 g_{\alpha 0} - \partial_\alpha g_{01}) \\
&= \frac{1}{2} \cancelto{0}{g^{30}}(\partial_0 g_{01} + \partial_1 g_{00} - \partial_0 g_{01}) + \frac{1}{2} \cancelto{0}{g^{31}}(\partial_0 g_{11} + \partial_1 g_{10} - \partial_1 g_{01}) \\
&+ \frac{1}{2} \cancelto{0}{g^{32}}(\partial_0 g_{21} + \partial_1 g_{20} - \partial_2 g_{01}) + \frac{1}{2} g^{33}(\partial_0 \cancelto{0}{g_{31}} + \partial_1 \cancelto{0}{g_{30}} - \partial_3 \cancelto{0}{g_{01}}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{20} = \Gamma^3_{02} &= \frac{1}{2} g^{3\alpha}(\partial_0 g_{\alpha 2} + \partial_2 g_{\alpha 0} - \partial_\alpha g_{02}) \\
&= \frac{1}{2}\cancelto{0}{g^{30}}(\partial_0 g_{02} + \partial_2 g_{00} - \partial_0 g_{02}) + \frac{1}{2}\cancelto{0}{g^{31}}(\partial_0 g_{12} + \partial_2 g_{10} - \partial_1 g_{02}) \\
&+ \frac{1}{2}\cancelto{0}{g^{32}}(\partial_0 g_{22} + \partial_2 g_{20} - \partial_2 g_{02}) + \frac{1}{2} g^{33}(\partial_0 \cancelto{0}{g_{32}} + \partial_2 \cancelto{0}{g_{30}} - \partial_3 \cancelto{0}{g_{02}}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{30} = \Gamma^3_{03} &= \frac{1}{2} g^{3\alpha}(\partial_0 g_{\alpha 3} + \partial_3 g_{\alpha 0} - \partial_\alpha g_{03}) \\
&= \frac{1}{2}\cancelto{0}{g^{30}}(\partial_0 g_{03} + \partial_3 g_{00} - \partial_0 g_{03}) + \frac{1}{2}\cancelto{0}{g^{31}}(\partial_0 g_{13} + \partial_3 g_{10} - \partial_1 g_{03}) \\
&+ \frac{1}{2}\cancelto{0}{g^{32}}(\partial_0 g_{23} + \partial_3 g_{20} - \partial_2 g_{03}) + \frac{1}{2} g^{33}(\cancelto{0}{\partial_0} g_{33} + \partial_3 \cancelto{0}{g_{30}} - \partial_3 \cancelto{0}{g_{03}}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{21} = \Gamma^3_{12} &= \frac{1}{2} g^{3\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) \\
&= \frac{1}{2}\cancelto{0}{g^{30}}(\partial_1 g_{02} + \partial_2 g_{01} - \partial_0 g_{12}) + \frac{1}{2}\cancelto{0}{g^{31}}(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}) \\
&+ \frac{1}{2}\cancelto{0}{g^{32}}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{12}) + \frac{1}{2} g^{33}(\partial_1 \cancelto{0}{g_{32}} + \partial_2 \cancelto{0}{g_{31}} - \partial_3 \cancelto{0}{g_{12}}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{31} = \Gamma^3_{13} &= \frac{1}{2} g^{3\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) \\
&= \frac{1}{2}\cancelto{0}{g^{30}}(\partial_1 g_{03} + \partial_3 g_{01} - \partial_0 g_{13}) + \frac{1}{2}\cancelto{0}{g^{31}}(\partial_1 g_{13} + \partial_3 g_{11} - \partial_1 g_{13}) \\
&+ \frac{1}{2}\cancelto{0}{g^{32}}(\partial_1 g_{23} + \partial_3 g_{21} - \partial_2 g_{13}) + \frac{1}{2} g^{33}(\partial_1 g_{33} + \partial_3 \cancelto{0}{g_{31}} - \partial_3 \cancelto{0}{g_{13}}) \\
&= \frac{1}{2}\left(\frac{1}{r^2 \sin^2\theta}\right)\left(2r\sin^2\theta\right) = \frac{1}{r}
\end{aligned}
$$

$$\Gamma^3_{32} = \Gamma^3_{23} = \frac{1}{2} g^{3\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23})$$

$$= \frac{1}{2} \cancelto{0}{g^{30}}(\partial_2 g_{03} + \partial_3 g_{02} - \partial_0 g_{23}) + \frac{1}{2} \cancelto{0}{g^{31}}(\partial_2 g_{13} + \partial_3 g_{12} - \partial_1 g_{23})$$

$$+ \frac{1}{2} \cancelto{0}{g^{32}}(\partial_2 g_{23} + \partial_3 g_{22} - \partial_2 g_{23}) + \frac{1}{2} g^{33}(\partial_2 g_{33} + \partial_3 \cancelto{0}{g_{32}} - \partial_3 \cancelto{0}{g_{23}})$$

$$= \frac{1}{2}\left(\frac{1}{r^2 \sin^2\theta}\right)\left(2r^2 \sin\theta\cos\theta\right) = \cot\theta$$

The thirteen non-zero terms are

$$\boxed{\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\frac{A'}{A}}$$

$$\boxed{\Gamma^1_{00} = \frac{1}{2}\frac{A'}{B}} \quad \boxed{\Gamma^1_{11} = \frac{1}{2}\frac{B'}{B}} \quad \boxed{\Gamma^1_{22} = -\frac{r}{B}} \quad \boxed{\Gamma^1_{33} = -\frac{r\sin^2\theta}{B}}$$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}} \quad \boxed{\Gamma^2_{33} = -\sin\theta\cos\theta}$$

$$\boxed{\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}} \quad \boxed{\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta}$$

5 Alternative Method (Lagrangian)

Did you get stressed whit a lot of calculations? An alternative way to obtain the non-zero connection coefficients is through the Euler-Lagrange equations. Often, this is useful because you get them without calculate one by one.

$$\boxed{L(x^\sigma, \dot{x}^\sigma) = \frac{1}{2} g_{\mu\nu}(x^\sigma)\dot{x}^\mu \dot{x}^\nu} \quad \boxed{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\sigma}\right) - \frac{\partial L}{\partial x^\sigma} = 0}$$

$$L = \frac{1}{2}\left(A\dot{t}^2 + B\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$$

$$\frac{\partial L}{\partial t} = 0, \qquad \frac{\partial L}{\partial \dot{t}} = A\dot{t}, \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = A'\dot{r}\dot{t} + A\ddot{t}$$

$$A'\dot{r}\dot{t} + A\ddot{t} = 0 \quad \Longrightarrow \quad \ddot{t} + \frac{A'}{A}\dot{r}\dot{t} = 0 \quad \longrightarrow \quad \text{geodesic}$$

$$\frac{\partial L}{\partial r} = \frac{1}{2}A'\dot{t}^2 + \frac{1}{2}B'\dot{r}^2 + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2, \qquad \frac{\partial L}{\partial \dot{r}} = B\dot{r}, \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = B'\dot{r}^2 + B\ddot{r}$$

$$B'\dot{r}^2 + B\ddot{r} - \frac{1}{2}A'\dot{t}^2 - \frac{1}{2}B'\dot{r}^2 - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = 0$$

$$\ddot{r} - \frac{1}{2}\frac{A'}{B}\dot{t}^2 + \frac{1}{2}\frac{B'}{B}\dot{r}^2 - \frac{r}{B}\dot{\theta}^2 - \frac{r}{B}\sin^2\theta\dot{\phi}^2 = 0 \quad \longrightarrow \quad \text{geodesic}$$

$$\frac{\partial L}{\partial \theta} = r^2\sin\theta\cos\theta\dot{\phi}^2, \qquad \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}, \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$$

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 = 0 \quad \Longrightarrow \quad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \quad \longrightarrow \quad \text{geodesic}$$

$$\frac{\partial L}{\partial \phi} = 0, \qquad \frac{\partial L}{\partial \dot{\phi}} = r^2\sin^\theta\dot{\phi}, \qquad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 2r\sin^2\theta\dot{r}\dot{\phi} + 2r\sin\theta\cos\theta\dot{\theta}\dot{\phi} + r^2\sin^\theta\ddot{\phi}$$

$$2r\sin^2\theta\dot{r}\dot{\phi} + 2r\sin\theta\cos\theta\dot{\theta}\dot{\phi} + r^2\sin^\theta\ddot{\phi} = 0 \quad \Longrightarrow \quad \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0 \quad \longrightarrow \quad \text{geodesic}$$

The geodesics are

$$\boxed{\ddot{t} + \frac{A'}{A}\dot{r}\dot{t} = 0} \qquad \boxed{\ddot{r} - \frac{1}{2}\frac{A'}{B}\dot{t}^2 + \frac{1}{2}\frac{B'}{B}\dot{r}^2 - \frac{r}{B}\dot{\theta}^2 - \frac{r}{B}\sin^2\theta\dot{\phi}^2 = 0}$$

$$\boxed{\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0} \qquad \boxed{\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0}$$

We can identify the non-zero terms

$$\boxed{\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\frac{A'}{A}}$$

$$\boxed{\Gamma^1_{00} = \frac{1}{2}\frac{A'}{B}} \quad \boxed{\Gamma^1_{11} = \frac{1}{2}\frac{B'}{B}} \quad \boxed{\Gamma^1_{22} = -\frac{r}{B}} \quad \boxed{\Gamma^1_{33} = -\frac{r\sin^2\theta}{B}}$$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}} \quad \boxed{\Gamma^2_{33} = -\sin\theta\cos\theta}$$

$$\boxed{\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}} \quad \boxed{\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta}$$

Note that when two different components are multiplying we divide it by 2. Also, in the second geodesic we reverse the sign for the time component since it is negative from our metric $(-+++)$.

6 Ricci Curvature Tensor

$$\boxed{R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\sigma} + \Gamma^{\sigma}_{\sigma\alpha}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\nu\alpha}}$$

$$R_{00} = \overset{\sigma=1}{\partial_\sigma \Gamma^{\sigma}_{00}} - \cancelto{0}{\partial_0} \Gamma^{\sigma}_{0\sigma} + \overset{\alpha=1}{\Gamma^{\sigma}_{\sigma\alpha}\Gamma^{\alpha}_{00}} - \overset{\alpha=\sigma=0,1}{\Gamma^{\alpha}_{0\sigma}\Gamma^{\sigma}_{0\alpha}}$$

$$= \partial_1 \Gamma^{1}_{00} + \overset{\sigma=0,1,2,3}{\Gamma^{\sigma}_{\sigma 1}\Gamma^{1}_{00}} - \Gamma^{0}_{01}\Gamma^{1}_{00} - \Gamma^{1}_{00}\Gamma^{0}_{01}$$

$$= \partial_1 \Gamma^{1}_{00} + \cancel{\Gamma^{0}_{01}\Gamma^{1}_{00}} + \Gamma^{1}_{11}\Gamma^{1}_{00} + \Gamma^{2}_{21}\Gamma^{1}_{00} + \Gamma^{3}_{31}\Gamma^{1}_{00} - \Gamma^{0}_{01}\Gamma^{1}_{00} - \cancel{\Gamma^{1}_{00}\Gamma^{0}_{01}}$$

$$= \partial_1 \Gamma^{1}_{00} + \Gamma^{1}_{11}\Gamma^{1}_{00} + \Gamma^{2}_{21}\Gamma^{1}_{00} + \Gamma^{3}_{31}\Gamma^{1}_{00} - \Gamma^{0}_{01}\Gamma^{1}_{00}$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{A'}{B}\right) + \left(\frac{1}{2}\frac{B'}{B}\right)\left(\frac{1}{2}\frac{A'}{B}\right) + \left(\frac{1}{r}\right)\left(\frac{1}{2}\frac{A'}{B}\right) + \left(\frac{1}{r}\right)\left(\frac{1}{2}\frac{A'}{B}\right) - \left(\frac{1}{2}\frac{A'}{A}\right)\left(\frac{1}{2}\frac{A'}{B}\right)$$

$$= \frac{1}{2}\frac{A''}{B} - \frac{1}{2}\frac{A'B'}{B^2} + \frac{1}{4}\frac{A'B'}{B^2} + \frac{A'}{Br} - \frac{1}{4}\frac{(A')^2}{AB}$$

$$= \frac{1}{2}\frac{A''}{B} - \frac{1}{4}\frac{A'B'}{B^2} + \frac{A'}{Br} - \frac{1}{4}\frac{(A')^2}{AB} \qquad / \cdot 4AB^2 r$$

$$= 2A''ABr - A'B'Ar + 4A'AB - (A')^2 Br$$

$$R_{11} = \overset{\sigma=1}{\partial_\sigma \Gamma^{\sigma}_{11}} - \overset{\sigma=0,1,2,3}{\partial_1 \Gamma^{\sigma}_{1\sigma}} + \overset{\alpha=1}{\Gamma^{\sigma}_{\sigma\alpha}\Gamma^{\alpha}_{11}} - \overset{\alpha=\sigma=0,1,2,3}{\Gamma^{\alpha}_{1\sigma}\Gamma^{\sigma}_{1\alpha}}$$

$$= \cancel{\partial_1 \Gamma^{1}_{11}} - \partial_1 \Gamma^{0}_{10} - \cancel{\partial_1 \Gamma^{1}_{11}} - \partial_1 \Gamma^{2}_{12} - \partial_1 \Gamma^{3}_{13} + \overset{\sigma=0,1,2,3}{\Gamma^{\sigma}_{\sigma 1}\Gamma^{1}_{11}} - \Gamma^{0}_{10}\Gamma^{0}_{10} - \Gamma^{1}_{11}\Gamma^{1}_{11} - \Gamma^{2}_{12}\Gamma^{2}_{12} - \Gamma^{3}_{13}\Gamma^{3}_{13} - \partial_1 \Gamma^{0}_{10}$$

$$= -\partial_1 \Gamma^{2}_{12} - \partial_1 \Gamma^{3}_{13} + \Gamma^{0}_{01}\Gamma^{1}_{11} + \cancel{\Gamma^{1}_{11}\Gamma^{1}_{11}} + \Gamma^{2}_{21}\Gamma^{1}_{11} + \Gamma^{3}_{31}\Gamma^{1}_{11} - \Gamma^{0}_{10}\Gamma^{0}_{10} - \cancel{\Gamma^{1}_{11}\Gamma^{1}_{11}} - \Gamma^{2}_{12}\Gamma^{2}_{12} - \Gamma^{3}_{13}\Gamma^{3}_{13}$$

$$= -\partial_1 \Gamma^{2}_{12} - \partial_1 \Gamma^{3}_{13} + \Gamma^{0}_{01}\Gamma^{1}_{11} + \Gamma^{2}_{21}\Gamma^{1}_{11} + \Gamma^{3}_{31}\Gamma^{1}_{11} - \Gamma^{0}_{10}\Gamma^{0}_{10} - \Gamma^{2}_{12}\Gamma^{2}_{12} - \Gamma^{3}_{13}\Gamma^{3}_{13}$$

$$= -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{A'}{A}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \left(\frac{1}{2}\frac{A'}{A}\right)\left(\frac{1}{2}\frac{B'}{B}\right) + \left(\frac{1}{r}\right)\left(\frac{1}{2}\frac{B'}{B}\right)$$

$$+ \left(\frac{1}{r}\right)\left(\frac{1}{2}\frac{B'}{B}\right) - \left(\frac{1}{2}\frac{A'}{A}\right)\left(\frac{1}{2}\frac{A'}{A}\right) - \left(\frac{1}{r}\right)\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)\left(\frac{1}{r}\right)$$

$$= -\frac{1}{2}\frac{A''}{A} + \frac{1}{2}\frac{(A')^2}{A^2} + \frac{\cancel{1}}{\cancel{r}^2} + \frac{\cancel{1}}{\cancel{r}^2} + \frac{1}{4}\frac{A'B'}{AB} + \frac{B'}{Br} - \frac{1}{4}\frac{(A')^2}{A^2} - \frac{\cancel{1}}{\cancel{r}^2} - \frac{\cancel{1}}{\cancel{r}^2}$$

$$= -\frac{1}{2}\frac{A''}{A} + \frac{1}{4}\frac{(A')^2}{A^2} + \frac{1}{4}\frac{A'B'}{AB} + \frac{B'}{Br} \qquad / \cdot 4A^2 Br$$

$$= -2A''ABr + (A')^2 Br + A'B'Ar + 4B'A^2$$

$$
\begin{aligned}
R_{22} &= \overset{\sigma=1}{\partial_\sigma \Gamma^\sigma_{22}} - \overset{\sigma=3}{\partial_2 \Gamma^\sigma_{2\sigma}} + \overset{\alpha=1}{\Gamma^\sigma_{\sigma\alpha}\Gamma^\alpha_{22}} - \overset{\alpha=\sigma=1,2,3}{\Gamma^\alpha_{2\sigma}\Gamma^\sigma_{2\alpha}} \\
&= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^3_{23} + \overset{\sigma=0,1,2,3}{\Gamma^\sigma_{\sigma 1}\Gamma^1_{22}} - \Gamma^1_{22}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\Gamma^3_{23} \\
&= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^3_{23} + \Gamma^0_{01}\Gamma^1_{22} + \Gamma^1_{11}\Gamma^1_{22} + \cancel{\Gamma^2_{21}\Gamma^1_{22}} + \Gamma^3_{31}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{21} - \cancel{\Gamma^2_{21}\Gamma^1_{22}} - \Gamma^3_{23}\Gamma^3_{23} \\
&= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^3_{23} + \Gamma^0_{01}\Gamma^1_{22} + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^3_{31}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{21} - \Gamma^3_{23}\Gamma^3_{23} \\
&= \frac{\partial}{\partial r}\left(-\frac{r}{B}\right) - \frac{\partial}{\partial \theta}(\cot\theta) + \left(\frac{1}{2}\frac{A'}{A}\right)\left(-\frac{r}{B}\right) + \left(\frac{1}{2}\frac{B'}{B}\right)\left(-\frac{r}{B}\right) + \cancel{\left(\frac{1}{r}\right)\left(-\frac{r}{B}\right)} \\
&\quad - \cancel{\left(-\frac{r}{B}\right)\left(\frac{1}{r}\right)} - (\cot\theta)(\cot\theta) \\
&= -\frac{1}{B} + \frac{B'r}{B^2} + (\csc^2\theta - \cot^2\theta) - \frac{1}{2}\frac{A'r}{AB} - \frac{1}{2}\frac{B'r}{B^2} \\
&= -\frac{1}{B} + \frac{1}{2}\frac{B'r}{B^2} + 1 - \frac{1}{2}\frac{A'r}{AB} \qquad / \cdot 2AB^2 \\
&= -2AB + B'Ar + 2AB^2 - A'Br
\end{aligned}
$$

$$
\begin{aligned}
R_{33} &= \overset{\sigma=1,2}{\partial_\sigma \Gamma^\sigma_{33}} - \cancelto{0}{\partial_3}\Gamma^\sigma_{3\sigma} + \overset{\alpha=1,2}{\Gamma^\sigma_{\sigma\alpha}\Gamma^\alpha_{33}} - \overset{\alpha=\sigma=1,2,3}{\Gamma^\alpha_{3\sigma}\Gamma^\sigma_{3\alpha}} \\
&= \partial_1 \Gamma^1_{33} + \partial_2 \Gamma^2_{33} + \overset{\sigma=0,1,2,3}{\Gamma^\sigma_{\sigma 1}\Gamma^1_{33}} + \overset{\sigma=3}{\Gamma^\sigma_{\sigma 2}\Gamma^2_{33}} - \Gamma^1_{33}\Gamma^3_{31} - \Gamma^3_{31}\Gamma^1_{33} - \Gamma^2_{33}\Gamma^3_{32} - \Gamma^3_{32}\Gamma^2_{33} \\
&= \partial_1 \Gamma^1_{33} + \partial_2 \Gamma^2_{33} + \Gamma^0_{01}\Gamma^1_{33} + \Gamma^1_{11}\Gamma^1_{33} + \Gamma^2_{21}\Gamma^1_{33} + \cancel{\Gamma^3_{31}\Gamma^1_{33}} + \cancel{\Gamma^3_{32}\Gamma^2_{33}} - \Gamma^1_{33}\Gamma^3_{31} - \cancel{\Gamma^3_{31}\Gamma^1_{33}} - \Gamma^2_{33}\Gamma^3_{32} - \cancel{\Gamma^3_{32}\Gamma^2_{33}} \\
&= \partial_1 \Gamma^1_{33} + \partial_2 \Gamma^2_{33} + \Gamma^0_{01}\Gamma^1_{33} + \Gamma^1_{11}\Gamma^1_{33} + \Gamma^2_{21}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^3_{31} - \Gamma^2_{33}\Gamma^3_{32} \\
&= \frac{\partial}{\partial r}\left(-\frac{r\sin^2\theta}{B}\right) + \frac{\partial}{\partial\theta}\left(-\sin\theta\cos\theta\right) + \left(\frac{1}{2}\frac{A'}{A}\right)\left(-\frac{r\sin^2\theta}{B}\right) + \left(\frac{1}{2}\frac{B'}{B}\right)\left(-\frac{r\sin^2\theta}{B}\right) \\
&\quad + \cancel{\left(\frac{1}{r}\right)\left(-\frac{r\sin^2\theta}{B}\right)} - \cancel{\left(-\frac{r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right)} - (-\sin\theta\cos\theta)(\cot\theta) \\
&= -\frac{\sin^2\theta}{B} + \frac{B'r\sin^2\theta}{B^2} - \cancel{\cos^2\theta} + \sin^2\theta - \frac{1}{2}\frac{A'r\sin^2\theta}{AB} - \frac{1}{2}\frac{B'r\sin^2\theta}{B^2} + \cancel{\cos^2\theta} \\
&= -\frac{\sin^2\theta}{B} + \frac{1}{2}\frac{B'r\sin^2\theta}{B^2} + \sin^2\theta - \frac{1}{2}\frac{A'r\sin^2\theta}{AB} \qquad / \cdot 2AB^2 \\
&= \left(-2AB + B'Ar + 2AB^2 - A'Br\right)\sin^2\theta \\
&= R_{22}\sin^2\theta
\end{aligned}
$$

The Ricci tensor components are

$$\boxed{R_{00} = 2A''ABr - A'B'Ar + 4A'AB - (A')^2Br}$$

$$\boxed{R_{11} = -2A''ABr + (A')^2Br + A'B'Ar + 4B'A^2}$$

$$\boxed{R_{22} = -2AB + B'Ar + 2AB^2 - A'Br} \qquad \boxed{R_{33} = R_{22}\sin^2\theta}$$

7 Find the Metric

Since $R_{\mu\nu} = 0$, then $R_{00} + R_{11} = 0$

$$0 = \cancel{2A''ABr} - \cancel{A'B'Ar} + 4A'AB - \cancel{(A')^2Br} - \cancel{2A''ABr} + \cancel{(A')^2Br} + \cancel{A'B'Ar} + 4B'A^2$$

$$0 = (4A)(A'B + AB')$$

$$0 = \frac{d}{dr}(AB)$$

$AB = C$

When we move far away from the black hole $r \to \infty$, space-time should look like Minkowski flat space-time.

$$A \to c^2, \quad B \to 1 \quad \implies \quad AB = c^2 \quad \implies \quad B = c^2A^{-1}$$

$$B = \frac{c^2}{A} \qquad B' = -c^2\frac{A'}{A^2}$$

Replacing B and B' into R_{22}

$$R_{22} = -2AB + B'Ar + 2AB^2 - A'Br$$

$$= -2A\left(\frac{c^2}{A}\right) + \left(-c^2\frac{A'}{A^2}\right)Ar + 2A\left(\frac{c^2}{A}\right)^2 - A'\left(\frac{c^2}{A}\right)r$$

$$= -2c^2 - c^2\frac{A'}{A}r + 2c^4\frac{1}{A} - c^2\frac{A'}{A}r$$

$$= -2c^2 - 2c^2\frac{A'}{A}r + 2c^4\frac{1}{A} \qquad / \cdot A$$

$$= -2c^2A - 2c^2A'r + 2c^4$$

$$= (-A - A'r + c^2)2c^2 = 0 \qquad \implies A'r = c^2 - A$$

Solving the ODE

$$\frac{dA}{dr}r = c^2 - A$$

$$\int \frac{1}{c^2 - A} dA = \int \frac{1}{r} dr$$

$$-\ln\left(c^2 - A\right) = \ln r + C$$

$$\left(c^2 - A\right)^{-1} = kr$$

$$c^2 - A = \frac{k}{r} \quad \Longrightarrow \quad A = c^2 - \frac{k}{r}$$

$$\boxed{A(r) = c^2\left(1 - \frac{k}{c^2 r}\right)} \qquad \boxed{B(r) = \left(1 - \frac{k}{c^2 r}\right)^{-1}}$$

The metric tensor turns out

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{k}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{k}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

8 Match with Newtonian Gravity

Why do we need to force the Schwarzschild to match with Newtonian gravity?

This relation ensures us that general relativity is consistent with classical Newtonian gravity where velocities are much smaller than the speed of light and gravity is relatively weak.

First, let's review some basic concepts.

$$\mathbf{F_g} = -\frac{GMm}{r^2}\mathbf{e}_r \qquad \mathbf{F} = -\nabla V = -\frac{dV}{d\mathbf{r}}$$

$$\mathbf{F} = -\frac{dV}{d\mathbf{r}} \quad \longrightarrow \quad dV = -\mathbf{F}d\mathbf{r} \quad \longrightarrow \quad V = -\int \mathbf{F}d\mathbf{r}$$

$$\begin{aligned} V(\mathbf{r}) &= GMm \int \frac{1}{r^2} d\mathbf{r} \\ &= GMm\left(-\frac{1}{r}\right) \\ &= -\frac{GMm}{r} \end{aligned}$$

V is the potential energy of a particle of a mass m in the presence of a mass M. Then, the gravitational field of the mass M is

$$\boxed{\Phi = -\frac{GM}{r}}$$

From Newton's Second Law

$$\mathbf{F} = m\mathbf{a}$$

$$\cancel{m}\mathbf{g} = \cancel{m}\mathbf{a}$$

$$\mathbf{g} = \mathbf{a}$$

$$-\nabla\Phi = \frac{d^2x^i}{dt^2}$$

Before solving for constant k, we need to consider:

1. **Low velocity**

   This means we use t as the path parameter and the four-velocity vector is $U^\mu = (c, 0, 0, 0)$.

2. **Weak gravity**

   This means we can approximate $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$.

First consideration allow us to write the geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0$$

As we want the time coordinate component g_{00}

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00}\cancelto{c}{\frac{dx^0}{dt}}\cancelto{c}{\frac{dx^0}{dt}} = 0$$

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00}c^2 = 0$$

Comparing these two equations

$$\frac{d^2x^i}{dt^2} + \nabla\Phi = 0 \qquad \frac{d^2x^i}{dt^2} + \Gamma^i_{00}c^2 = 0$$

we can deduce that

$$\Gamma^i_{00} = \frac{1}{c^2}\nabla\Phi = \frac{1}{c^2}\frac{\partial\Phi}{\partial x^i} = \frac{1}{c^2}\partial_i\Phi$$

Following the second consideration

$$\Gamma^i_{00} = \frac{1}{2} g^{i\alpha}(\cancelto{0}{\partial_0 g_{\alpha 0}} + \cancelto{0}{\partial_0 g_{\alpha 0}} - \partial_\alpha g_{00})$$

$$= \frac{1}{2}\eta^{i\alpha}(-\partial_\alpha h_{00})$$

$$= -\frac{1}{2}\cancelto{1}{\eta^{ii}}(\partial_i h_{00})$$

$$= -\frac{1}{2}\partial_i h_{00}$$

Solving for h_{00}

$$\frac{1}{c^2}\cancel{\partial_i}\Phi = -\frac{1}{2}\cancel{\partial_i}h_{00} \implies h_{00} = -\frac{2\Phi}{c^2}$$

Our approximation gives us

$$g_{00} \approx \eta_{00} + h_{00}$$

$$\approx -1 - \frac{2\Phi}{c^2}$$

$$\approx -\left(1 + \frac{2\Phi}{c^2}\right)$$

From the metric tensor calculated in the last section we can obtain the constant k.

$$1 - \frac{k}{c^2 r} = 1 + \frac{2\Phi}{c^2}$$

$$\cancel{1} - \frac{k}{\cancel{c^2}\cancel{r}} = \cancel{1} - \frac{2GM}{\cancel{c^2}\cancel{r}}$$

$$k = 2GM$$

Finally, we get the Schwarzschild metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

or

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$